The Reconstruction of the Relative Phases and Polarization of the Electromagnetic Field Based on Amplitude Measurements

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Abstract—**This paper addresses the problem of determining the relative phases among the components of a monochromatic electromagnetic field at a single point based on amplitude measurements only. This problem arises in many practical situations, from the determination of the polarization characteristics of an antenna in its near field to the design of electromagnetic field probes that provide phase information without the need for synchronization leads. In this paper, we answer the following questions: 1) How is the phase information reconstruction from amplitude measurements? 2) Do amplitude measurements contain enough information to reconstruct the phases uniquely? 3) How many amplitude measurements are required? and 4) In what directions must the amplitude measurements be taken to ensure that the reconstructed phases are unique? To answer these questions, we propose and solve a more general problem in n dimensions, the special case where $n = 3$ being the solution to the problem above.**

I. Introduction

In MANY practical situations, one needs to know the relative phases among the components of a sinusoidally time-varying vector, in addition to their amplitudes. The determination of the polarization characteristics of an antenna, for example, requires that the relative phases as well as the amplitudes be measured. A commonly used procedure is to find the direction along which the projected amplitude is zero; the polarization ellipse then lies in the plane perpendicular to this direction. This procedure requires many measurements. Further measurements are needed in this plane to determine the characteristics of the ellipse. The relative phases are obtained using a synchronizing lead to provide a reference time-base; the relative phases are then computed from time-delay measurements. These measurements pose practical problems at the high frequency limit. Relative phases among the components of a sinusoidally time-varying vector are also needed in determining the source direction of plane waves, and in these situations, there are no synchronizing leads. Measuring time delay of the components with respect to a selected one is unreliable in practice. In quantum systems, the amplitudes are related to the probability of finding a particular value of a quantum number such as spin in a particular direction in 3-space. The relative phases in this case give the relative orientations of different states in the system's Hilbert space and give rise to interference behavior. In this extreme example, the relative phases cannot be measured directly (there is no quantum "synchronizing lead") but the probabilities can be.

Motivated by these and other practical considerations, we formulate and solve a general problem of relative-phase reconstruction using amplitude measurements only. In this paper, we answer the following questions: 1) How is the phase information reconstructed from amplitude measurements? 2) Do amplitude measurements contain enough information to reconstruct the phases uniquely? 3) How many amplitude measurements are required? and 4) In what directions must the amplitude measurements be taken to ensure that the reconstructed phases are unique? Question 3) has no obvious answer because the relative-phase reconstruction from amplitudes is a non-linear process. Thus, in n dimensions, there are $(n - 1)$ relative phases among the components of a sinusoidally time-varying vector, but one cannot "automatically" conclude that one needs $(n - 1)$ amplitudes (which is clearly false for two dimensions: *three* amplitude measurements are needed to determine *one* relative phase between the *two* components of the vector). The problem is as follows.

A real vector X in n dimensions whose components are sinusoidally varying with time at a fixed frequency can be represented by an n-dimensional complex vector $Ze^{j\omega t}$ whose real part is X. Each of the n components of Z is a complex number with an amplitude and a phase angle. In different real coordinate systems, the components of X transform linearly, but the components of Z are not transformed by the same change-of-coordinate transformation, and the amplitudes of Z transform nonlinearly in general. Only when X has linear polarization do the components of Z transform linearly (using in that case the same linear transformation for the components of X). For example, in two dimensions, a real vector $X = (5 \cos \omega t, 3 \sin \omega t)$ is represented by $Z = (5, 3e^{-j\pi/2})$. In another coordinate

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system rotated by 90 degrees, the linear transformation is

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Under this transformation, X becomes $X' = XL = (3 \sin \omega t, -5 \cos \omega t)$ with complex representation $Z' = (3e^{-j\pi/2}, 5e^{j\pi})$. Clearly, the amplitudes of the components of Z and Z' are not related by the same linear transformation L. It will be shown subsequently that the amplitudes of the components of Z transform non-linearly in general. It is because of this nonlinearity that the phase information can be retrieved from amplitude measurements in different directions.

For the special case $n = 1$, where polarization is not significant, the problem has been solved, and found to have practical applications [1]–[4]. These solutions, however, have no immediate generalization to dimensions higher than 1. It is desirable to have the solution for the case where the dimension n is a positive integer, and in particular, for the case where $n = 3$.

This paper describes a method to reconstruct the relative phases of the components of Z (and therefore of X) based on the amplitude measurements of its n components in an orthogonal coordinate system and at most $n(n - 1)/2$ additional amplitude measurements in different directions, making a total of at most $n(n + 1)/2$ measurements. The authors will show the necessary and sufficient condition on these additional directions to ensure uniqueness of the phase and polarization reconstruction for any arbitrary vector X. The result of this phase reconstruction is the complete characterization of the polarization of X except for chirality. That is, with amplitude information, we can determine the polarization ellipse, but not the sense.

II. The Formulation of the Problem

Let X be a real vector in n dimensions whose components are sinusoidally varying with time and have arbitrary (unknown) phases. Let $A_1, \cdots, A_n$ be the amplitudes of the components of X in one coordinate system, and let $B_1, \cdots, B_k$ be the amplitudes of the projection of X in k additional directions described by unit vectors $N_1, N_2, \cdots, N_k$. The problem is to find the relative phases among the components of X and characterize the polarization (i.e. the time-dependent behavior of X). How many amplitude measurements $B_1, \cdots, B_k$ in addition to the $A_1, \cdots, A_n$ are required to reconstruct the polarization uniquely? How does one choose the unit vectors $N_1, N_2, \cdots, N_k$ to ensure unique reconstruction of the polarization ellipse for arbitrary X?

The solution to the above problem is concisely stated in the next section after the formulation of some preliminary concepts and the solutions to the following intermediate problems.

1. The Polarization of a Complex Vector in n Dimensions: Let Z be a complex vector in n dimensions whose components are arbitrary complex numbers. Let X be defined by

$$X = \text{Re}\,\{Ze^{j\omega t}\}.$$

The polarization of Z is defined by the time-dependent behavior of X. It has been shown [5] that for an arbitrary finite integer n greater than 1, and for any arbitrary complex vector Z, the locus traced in time by X must be a two dimensional ellipse (with circle and straight line segment being degenerate cases) centered at the origin. (This result was also obtained for the special case where $n = 3$ by Deno and Zaffanella [6] in 1978). The major and minor axes of the ellipse are described by the following procedure:

Let the complex vector Z with components Z_i be

$$Z_i = A_i \exp(j\varphi_i).$$

The semi-major axis V and semi-minor axis R of the ellipse have components V_i and R_i defined by [5]

$$V_i = A_i \cos(\varphi_i - \psi) \quad (1)$$

and

$$R_i = A_i \sin(\varphi_i - \psi) \quad (2)$$

where

$$\psi = \tfrac{1}{2} \arg(Z^t Z).$$

Since the polarization has been shown [5] to be two-dimensional for arbitrary n greater than 1 and for arbitrary Z, the handedness (or chirality) of the polarization is not unique given the amplitude information for n greater than 2. This is because in spaces of dimension three or higher, a two-dimensional ellipse with a clockwise sense on the boundary will become one with a counterclockwise sense on the boundary when viewed from the other side. Only in a two dimensional world is there no "other side," and chirality in that case is theoretically meaningful. This explains why the reconstructed polarization described later is complete except for chirality in general.

In summary, when $n > 1$, the time-dependent real vector $X = \text{Re}\,\{Ze^{j\omega t}\}$ traces out a two-dimensional ellipse with its major and minor axes described by the two real vectors V and R.

2. The Magnitude of the Sinusoidally Time-Varying Projection of X in an Arbitrary Direction: Let Z be the complex vector related to X by $X = \text{Re}\,\{Ze^{j\omega t}\}$ as before. Let V and R be the major and minor axes of the ellipse constructed from Z as described previously. Let the arbitrary direction be specified by a unit vector N. The amplitude and phase of the projection of X in the direction N can be found as follows:

$$X(u) = V \cos u + R \sin u, \qquad 0 \le u \le 2\pi.$$

Projected on the direction N, this ellipse becomes

$$X_N(u) = (V, N) \cos u + (R, N) \sin u, \qquad 0 \le u \le 2\pi,$$

where (V, N) denotes the dot product of V and N. The maximum value of X_N, denoted by C, is

$$C = \left| (V, N) \cos \tan^{-1} \left(\frac{(R, N)}{(V, N)} \right) + (R, N) \sin \tan^{-1} \left(\frac{(R, N)}{(V, N)} \right) \right|,$$

which becomes

$$C = \sqrt{\{(V, N)\}^2 + \{(R, N)\}^2}. \tag{3}$$

This is the amplitude of the projection of X in the direction N.

In summary, given a real vector X whose components are sinusoidally varying with time, and a unit vector N, one can compute the magnitude of the (sinusoidally time-varying) projection of X in the direction N. Or more generally, given k directions specified by the unit vectors N_1, $N_2, \cdots, N_k$, the above analysis produces the magnitudes $C_1, C_2, \cdots, C_k$ of the projections of X in the k directions.

III. The Solution to the Problem of Phase and Polarization Reconstruction

Let A be a real vector with components $A_1, \cdots, A_n$ which are the amplitudes of the sinusoidally time-varying components of a real vector X in one coordinate system. Let B be a real vector with components $B_1, \cdots, B_k$ which are the amplitudes of the components of X in k directions specified by unit vectors $N_1, N_2, \cdots, N_k$.

We will solve the phase and polarization reconstruction problem using nonlinear optimization as follows: We begin by guessing the relative phases $\varphi_2, \varphi_3, \cdots, \varphi_n$ of the components of X relative to φ_1, which is set equal to zero. These phases are combined with the given amplitudes $A_1, \cdots, A_n$ to form a complex vector Z'. To facilitate the statement of the solution to the phase and polarization reconstruction, one formalizes this construction of Z' by defining a function K_A as follows:

$$K_A \begin{pmatrix} 0 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{pmatrix} = Z' = \begin{pmatrix} A_1 e^{j0} \\ A_2 e^{j\varphi_2} \\ \vdots \\ A_n e^{j\varphi_n} \end{pmatrix}.$$

From this complex vector Z', one computes the real vectors V and R as described in Section II-1. Using these two vectors and the k given unit vectors $N_1, N_2, \cdots, N_k$, one computes the magnitudes $C_1, C_2, \cdots, C_k$ according to Section II-2. Let C be a vector whose components are the magnitudes $C_1, C_2, \cdots, C_k$. This vector C is compared with the given vector B, and the error, denoted by $H_B(C)$, is

$$H_B(C) = \left[\sum_{m=1}^{k} (B_m - C_m)^2 \right]^{1/2}.$$

A solution to the Phase and Polarization Reconstruction Problem is found by nonlinear minimization of the composite function $L = H_B \circ K_A$ over the $\varphi_2, \varphi_3, \cdots, \varphi_n$ for a given A and B. Theoretically, with no errors in the specification of the amplitudes A and B, the minimum value of L would be precisely zero. In practice, when A and B are experimentally measured, the minimum value of L, not necessarily zero, is called the Residual. It serves as an indication of how well the vectors A and B actually describe a physically realizable polarization state for some vector X.

IV. The Necessary and Sufficient Condition for the Uniqueness of the Phase and Polarization Reconstruction

It is desirable to know whether or not the solution depends on the initial choice of phases. The purpose of this section is to answer this question by finding the necessary and sufficient condition for the uniqueness of the reconstruction in n dimensions. This condition will dictate the choice of the directions in which the amplitudes are measured.

Let Q be a complex vector whose real part is V and whose imaginary part is R. That is, $Q = V + jR$. Equations (3) now becomes $C = |(Q, N)|$, or more generally, for different unit vectors $N_1, N_2, \cdots, N_k$,

$$C_m = |(Q, N_m)|. \tag{4}$$

From (1) and (2) defining V and R, one has

$$Q = V + jR = \begin{pmatrix} A_1 e^{j(\varphi_1 - \psi)} \\ A_2 e^{j(\varphi_2 - \psi)} \\ \vdots \\ A_n e^{j(\varphi_n - \psi)} \end{pmatrix} = Ze^{-j\psi}.$$

Substituting this expression into (4), one obtains

$$C_m = |(Z, N_m)|.$$

Therefore,

$$C_m^2 = (Z, N_m)^* (Z, N_m), \tag{5}$$

where the superscript * denotes the complex conjugation of the preceding quantity.

Since this section is concerned with the uniqueness and not the existence of the solution, let us assume that at least one solution exists. That is, for a given set of amplitudes $A_1, A_2, \cdots, A_n, B_1, B_2, \cdots, B_k$, and a given set of unit vectors $N_1, N_2, \cdots, N_k$, there exists a set of phases such that the computed magnitudes C_m in (4) are equal to the given magnitudes B_m, $m = 1, 2, \cdots, k$. Equivalently, there exists at least one complex vector Z whose components have the given magnitudes $A_1, A_2, \cdots, A_n$ such that

$$B_m^2 = (Z, N_m)^*(Z, N_m), \qquad m = 1, 2, \cdots, k. \tag{6}$$

Given one solution to (6), one can always obtain other solutions by complex conjugating Z, by multiplying Z by

$e^{j\delta}$ for any arbitrary real δ, or by adding a multiple of 2π to any one of the phases. First, complex conjugating Z causes all phases of its components to change sign. Equations (1) and (2) show that changing the signs of the phases does not affect the vector V but changes the vector R to $-R$. However, this does not change the ellipse defined by V and R, and hence does not change the polarization of Z except for chirality which is not considered in this paper (see Introduction). Secondly, multiplying Z by $e^{j\delta}$ for any arbitrary real δ does not affect the relative phases of the components and consequently does not affect the polarization. Neither does the addition of an arbitrary multiple of 2π to any one of the phases since doing so does not change V or R.

Therefore, any two solutions to the phase and polarization reconstruction problem may be considered the same if the phases differ by a plus-minus sign, by the same additive constant, or by arbitrary multiple of 2π. For the remaining of this section, all ambiguities of this nature are removed by requiring that the phase of the first component of Z be zero, the phase of the second component of Z be between 0 and π inclusively, and all other phases be non-negative and less than 2π. A solution to (6) satisfying these is said to be in *standard form*. The question of uniqueness is, thus, about the existence of different standard-form solutions to (6).

Letting Z^+ denotes the transposed complex conjugate of the vector Z and N_m^t denote the transpose of the vector N_m, (6) becomes

$$B_m^2 = Z^+ N_m N_m^t Z, \qquad m = 1, 2, \cdots, k. \tag{7}$$

Writing out (7) in full, one obtains, for $m = 1, 2, \cdots, k$,

$$\begin{aligned} B_m^2 = {} & z_1^* z_1 (N_m)_1 (N_m)_1 + z_1^* z_2 (N_m)_1 (N_m)_2 \\ & + \cdots + z_1^* z_n (N_m)_1 (N_m)_n + z_2^* z_1 (N_m)_2 (N_m)_1 \\ & + z_2^* z_2 (N_m)_2 (N_m)_2 + \cdots + z_2^* z_n (N_m)_2 (N_m)_n \\ & + \cdots + z_n^* z_1 (N_m)_n (N_m)_1 + z_n^* z_2 (N_m)_n (N_m)_2 \\ & + \cdots + z_n^* z_n (N_m)_n (N_m)_n, \end{aligned} \tag{8}$$

where $(N_m)_p$ denotes the pth component of the vector N_m. Of the terms on the right-hand side of (8), the vector N_m is given, and $z_p^* z_p$ is A_p^2, the squared amplitude of the pth component of Z, which is also given. Thus, only those terms involving $z_p^* z_q$ with $p \neq q$ contain the unknown relative phases and hence are unknown. Since Re $\{z_p^* z_q\}$ = Re $\{z_q^* z_p\}$, one may take the real part of both sides of (8), bring known (given) terms to the left-hand side, and rearrange to yield

$$B_m^2 - \sum_{p=1}^{n} (N_m)_p^2 A_p^2 = \sum_{1 \le p < q \le n} [2(N_m)_p (N_m)_q] \operatorname{Re}\{z_p^* z_q\}. \tag{9}$$

Let F_m denote the left-hand side. Let $D_{m,pq}$ denote the factor in the square bracket, and let

$$\xi_{pq} = \operatorname{Re}\{z_p^* z_q\}, \qquad 1 \le p < q \le n. \tag{10}$$

Then, for each $m = 1, 2, \cdots, k$, (9) can be written

$$F_m = \sum_{1 \le p < q \le n} D_{m,pq} \xi_{pq}. \tag{11}$$

Finally, one can assemble the F_m's into a k-tuple F, assemble the ξ_{pq}'s into a $n(n-1)/2$-tuple ξ, and let D be a real matrix with k rows and $n(n-1)/2$ columns indexed by pairs of integers pq with $p < q$. Let the element of the matrix D located at row m and column pq be $D_{m,pq}$. Then, (11) can be written in matrix form

$$D\xi = F. \tag{12}$$

In order for (12) to have a unique solution, it is necessary (but not sufficient) that the number k of rows of the matrix D be equal to the number $n(n-1)/2$ of columns. Thus, it is necessary that

$$k = \frac{n(n-1)}{2}. \tag{13}$$

Furthermore, there are two possible cases: det $(D) \neq 0$ or det $(D) = 0$.

Case 1: det $(D) \neq 0$

Since the determinant of D is not zero, D is invertible, and (12) has a *unique* solution

$$\xi = D^{-1} F \tag{14}$$

which uniquely determines the ξ_{pq}. Using the defining (10) for the ξ_{pq}, one obtains

$$\xi_{pq} = \operatorname{Re}\{z_p^* z_q\} = A_p A_q \cos(\varphi_p - \varphi_p). \tag{15}$$

Since a solution exists by hypothesis, the ξ_{pq} computed from (14) must have a magnitude less than or equal to $A_p A_q$ (because otherwise (15) has no real solution for the φ_p and φ_q, contrary to hypothesis). Therefore, (15) can be solved to obtain

$$\varphi_q - \varphi_p = \pm \eta_{qp}, \quad \text{where} \quad 0 \le \eta_{qp} < 2\pi. \tag{16}$$

By definition of the ξ_{pq}, p is strictly less than q, and $\varphi_1 = 0$ by construction. Therefore, with $p = 1$, (16) gives

$$\varphi_q = \pm \eta_{q1} \equiv \pm \eta_q, \qquad q = 2, \cdots, n, \tag{17}$$

where

$$0 \le \eta_q < 2\pi.$$

On the surface, (17) suggests that there are 2^{n-1} possible combinations of phases, but this is not true because of the constraints imposed by (16) as follows: For each pair (p, q), one computes the left-hand-side of (16) using (17). This produces the following possibilities:

1. $\varphi_q = \eta_q$, $\varphi_p = \eta_p$. Then $\varphi_q - \varphi_p = (\eta_q - \eta_p)$.
2. $\varphi_q = \eta_q$, $\varphi_p = -\eta_p$. Then $\varphi_q - \varphi_p = (\eta_q + \eta_p)$.

3. $\varphi_q = -\eta_q, \varphi_p = \eta_p$. Then $\varphi_q - \varphi_p = -(\eta_q + \eta_p)$.

4. $\varphi_q = -\eta_q, \varphi_p = -\eta_p$. Then $\varphi_q - \varphi_p = -(\eta_q - \eta_p)$.

A comparison of these with the right-hand-side of (16) shows that either $\eta_{qp} = (\eta_q + \eta_p)$ or $\eta_{qp} = |\eta_q - \eta_p|$ *but not both*. Therefore, a given value of η_{qp} produced from (16) determines either Possibilities 1 and 4 or else Possibilities 2 and 3. In either case, the φ_p and φ_q *both* change sign. As discussed previously at the beginning of this section, this sign change does not lead to different standard-form solutions to the phase and polarization reconstruction problem.

Thus, if det $(D) \neq 0$, the reconstructed phases in standard form and the reconstructed polarization are unique. Or equivalently, det $(D) \neq 0$ is a *sufficient* condition for the uniqueness of the polarization and the uniqueness of the phases in standard form.

Case 2: det $(D) = 0$.

Since det $(D) = 0$, D is singular. Therefore, it has a nonzero kernel with a dimension greater than zero. Let $y_1, y_2, \cdots, y_s$ be a basis for the kernel of D where s is a positive integer less than or equal to $n(n-1)/2$. The solution to (12) is no longer unique. The general solution has the following form:

$$\xi = \xi^0 + g_1 y_1 + g_2 y_2 + \cdots + g_s y_s,$$

where ξ^0 is a any solution to (12), and $g_1, g_2, \cdots, g_s$ are arbitrary real numbers. Proceeding as in Case 1, one obtains

$$\varphi_q - \varphi_p = \pm\eta_{qp}(g_1, g_2, \cdots, g_s)$$

and

$$\varphi_q = \pm\eta_{q1} = \pm\eta_q(g_1, g_2, \cdots, g_s)$$

for some real *functions* η_{qp} and η_q of the $g_1, g_2, \cdots, g_s$. Reasoning exactly as in Case 1 shows that for a *fixed* set of $g_1, g_2, \cdots, g_s$, the phases are unique up to a simultaneous sign change which leads to the same standard-form solution. However, a *different* set of the $g_1, g_2, \cdots, g_s$ would clearly produce a *different* standard-form solution.

Thus, if det $(D) = 0$, the reconstructed phases in standard form and the reconstructed polarization are not unique. Or equivalently, det $(D) \neq 0$ is a *necessary* condition for the uniqueness of the polarization and the uniqueness of the phases in standard form.

The conclusions of Case 1 and Case 2 together state that det *(D) $\neq 0$ is a necessary and sufficient condition for the uniqueness of the reconstructed polarization and the uniqueness of the reconstructed phases in standard form.* As the directions $N_1, N_2, \cdots, N_k$ are used in computing the matrix D, the condition above is a necessary and sufficient condition for choosing these directions along which additional amplitude measurements are taken.

Finally, for an arbitrary finite positive integer n, it is always possible to find the direction vectors $N_1, N_2, \cdots, N_k$, where $k = n(n-1)/2$, such that the matrix D has nonzero determinant because one can always pick those vectors as follows. Because $k = n(n-1)/2$, one can index these vectors by a double subscript (p, q) where $1 \le p < q \le n$. Let N_{pq} be chosen to be the vector whose pth and qth components are both equal to $(1/\sqrt{2})$ and all other components are zero. Computing the elements $D_{m,pq}$ of matrix D using the above vectors N_{pq} is described previously, one can verify that the matrix D is the k-by-k identity matrix whose determinant is 1. Therefore, by the necessary and sufficient condition for uniqueness, the phase and polarization reconstruction using these direction vectors will be unique.

The explicit construction of the vectors N_{pq} shows that the necessary and sufficient condition det $D \neq 0$ can actually be satisfied.

V. Examples of the Phase and Polarization Reconstruction and its Uniqueness

Although the following results were obtained by nonlinear minimization using the standard method of gradient [7], the details of the minimization are not too important for the purposes of this paper and will be described only briefly. The partial derivatives are computed using central differencing with the provision for saddle points and minima. The step size in the direction of the negative of the gradient in the space of phase angles is initially one degree. This step size is halved each time the scalar product of the gradient at the m step with the gradient at the $(m-1)$ step is negative, which signals that the local minimum has been approached and passed. A minimum is obtained when the Euclidean length of the gradient vector is less than 1.0×10^{-5} times the value of the largest given amplitude. The starting guesses for the phase angles in degrees are $\varphi_1 = 0$, $\varphi_2 = 10$, and $\varphi_3 = 10$. If these starting guesses do not produce a residual of less than 0.1, then the minimization starts over with new guesses, where φ_2 and φ_3 are increased by 45 degrees. This is repeated 6 more times if necessary.

Example 1: In three dimensions, $n = 3$, $n(n+1)/2 = 6$, and one needs a total of six amplitude measurements: three measurements along the axes of an orthogonal coordinate system, and three more in three directions specified by unit vectors N_1, N_2, and N_3. Let these unit vectors be as follows:

$$N_1 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}, \quad N_2 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \quad N_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

That is, these vectors are obtained by rotating the old coordinate axes by 60 degrees around the vector (1, 1, 1), counter-clockwise if viewed in the direction from (1, 1, 1) to the origin.

To determine whether these choices of the three additional directions will lead to uniqueness of the phase and polarization reconstruction, one applies the necessary and sufficient conditions derived in the previous section. The matrix D is

$$D = \frac{4}{9}\begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix},$$

which has zero determinant. Therefore, any phase and polarization reconstruction using amplitude measurements along the N_1, N_2, and N_3 directions will not be unique. This is confirmed by the following numerical calculations:

Let the amplitude measurements be 6, 4, 5, along an orthogonal coordinate system, and let the additional amplitude measurements along N_1, N_2, and N_3 be 5.8897, 5.2202, and 3.8808. The reconstruction using these six numbers produces two different standard-form solutions: one with phases in degrees being 0, 45.0000, 90.0000, and the other, 0, 104.2940, and 220.2520.

Example 2: At a particular point in three dimensional space, a monochromatic elliptically polarized electromagnetic plane wave is moving with the phase-velocity vector in the direction (1, 1, 1). Suppose that the semi-major axis of the ellipse is in the direction (1, 0, −1), where the amplitude of the electric field is 5 millivolts/meter, and the semi-minor axis is in the direction (−1, 2, −1), where the amplitude is 3 millivolts/meter. (This wave is easily physically realizable by appropriately placing and oriented two mutually perpendicular sinusoidally-driven dipole antennas with driving amplitudes in the ratio 5 to 3 and 90 degrees out of phase).

By an elementary computation, the electric field is found to have the following amplitudes:

$$A_x = 3.742 \text{ mV/m},$$

$$A_y = 2.449 \text{ mV/m, and}$$

$$A_z = 3.742 \text{ mV/m}.$$

Compared to the x-component, the y-component has a phase lead of 109.1 degrees, and the z-component has a phase lead of 218.2 degrees. Along three additional directions specified by the vectors

$$N_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad N_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

the electric field has amplitudes

$$B_1 = 2.646 \text{ mV/m},$$

$$B_2 = 1.732 \text{ mV/m, and}$$

$$B_3 = 2.646 \text{ mV/m}.$$

To illustrate the method of relative-phase reconstruction in this paper, we will use only the amplitude information given above and derive everything else.

First, from the three vectors N_1, N_2, and N_3, one computes the matrix D and finds that the determinant of D is nonzero. The necessary and sufficient condition in Section IV states that the reconstructed phases in standard form will be unique. Next, using the six amplitudes along the x-, y-, and z-directions as well as along the three additional directions, the method of non-linear optimization described previously yields, 0, 109.1, and 218.2 degrees for the relative phases in standard form among the cartesian components of the electric field. Using these phases, the given amplitudes, and (1) and (2), one finds that the electric field reaches a maximum of 5 millivolts/meter in the direction of the vector $V = (1, 0, -1)$, and a minimum of 3 millivolts/meter in the direction of the vector $R = (-1, 2, -1)$. The cross-product of V and R gives the direction, up to a plus or minus sign, of the phase velocity, which is $\pm(1, 1, 1)$. Finally, the knowledge of the location of the source of the electromagnetic wave removes the sign ambiguity.

A comparison with the "given" information in the first two paragraphs of this example shows that from only the six amplitudes, the method of relative-phase reconstruction has correctly produced all other information, including the correct relative phases.

VI. Conclusion

From only amplitude measurements, it is possible to reconstruct uniquely the phases in standard form and the polarization ellipse of a vector whose components are sinusoidally time-varying in an n-dimensional space where n is a finite integer greater than 1. This unique reconstruction will require at most $n(n + 1)/2$ amplitude measurements, n of which are along the axes of an orthogonal coordinate system. The remaining $n(n - 1)/2$ amplitude measurements are along $n(n - 1)/2$ additional directions which must satisfy a necessary and sufficient condition in order that the reconstructed phases in standard form and the reconstructed polarization ellipse are unique. These additional directions can always be found to satisfy this condition for any finite positive integer n greater than 1. The special case where $n = 3$ is applicable to the relative phase and polarization reconstructions of the electromagnetic field from amplitude measurements.

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